ঐতিহাসিক সময়কাল

Historical Timeline

There is a long history of prime numbers from Euclid to Bernhard Riemann and even until today (2020). In addition, there is almost three hundred years of history surrounding the Riemann hypothesis and the prime number theorem. While the authoritative history of these ideas has yet to appear, this timeline briefly summarizes the historic prime points.

300BC Euclid first proves that there are infinitely many prime numbers.

1737 Euler proves the Euler product formula (for real s > 1),

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

1738 Euler invents the “Euler-Maclaurin summation method”.

1742 Maclaurin invents the “Euler-Maclaurin summation method”.

1742 Goldbach proposes his conjecture to Euler in a letter.

1792 Gauss proposes what would later become the prime number theorem.

1802 Haros discovers and proves the results concerning the general properties of Farey series.

1845 Bertrand postulates that for a > 1 there is always a prime that lies between *a* and *2a*.

1850 Chebyshev proves Bertrand’s postulate using elementary methods.

1859 Bernhard Riemann publishes his ***Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*** ("On the Number of Primes Less Than a Given Magnitude"), in which he proposes the Riemann hypothesis. The original statement is as follow:

"*One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.*"

Riemann's paper contains the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

His work contains analysis of the Reimann zeta function. Some of Riemann's analysis is overlooked until it is reinvigorated by Siegel in 1932.

1885 Stieltjes claims to have a proof of what would later be called Mertens' conjecture. His proof is never published, nor found among his papers posthumously. Mertens' conjecture implies the Riemann hypothesis. This is perhaps the first significant failed attempt at a proof.

1890 Some time after this date, Lindelöf proposed the Lindelöf hypothesis. This conjecture concerns the distribution of the zeros of Riemann's zeta function. It is still unproven.

1896 Hadamard and de la Vallée Poussin independently prove the prime number theorem. The proof relies on showing that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$.

1897 Mertens publishes his conjecture. (This conjecture is proved incorrect by Odlyzko and te Riele in 1985.)

1900 David Hilbert publishes a list of 23 unsolved problems in the Second International Congress of Mathematicians in Paris on the morning of Wednesday, August 8, 1900. The Riemann Hypothesis was the eighth. Then he indirectly warranted that no body could solve the Riemann hypothesis within next 1000 years.

1901 Von Koch proves that the Riemann hypothesis is equivalent to the statement

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(\sqrt{x}\log x).$$

1903 Gram calculates the first 15 zeros of $\zeta(s)$ on the critical line.

1903 In the decade following 1903, Gram, Backlund, and Hutchinson independently apply Euler–Maclaurin summation to calculate $\zeta(s)$ and to verify the Riemann hypothesis for $t \leq 300$ (where $s = ½ + it$).

1912 Littlewood proves that Mertens' conjecture implies the Riemann hypothesis.

1912 Littlewood proves that $\pi(n) < \mathrm{Li}(n)$ fails for some n.

1912 Backlund develops a method of determining the number of zeros of $\zeta(s)$ in the critical strip $0 < \Re(s) < 1$ up to a given height. This method is used through 1932.

1914 Hardy proves that infinitely many zeros of $\zeta(s)$ lie on the critical line.

1914 Backlund calculates the first 79 zeros of $\zeta(s)$ on the critical line.

1914 Littlewood proves that $\pi(n) < \mathrm{Li}(n)$ fails for infinitely many n.

1914 Bohr and Landau prove that if $N(\sigma, T)$ is the number of zeros of $\zeta(s)$ in the rectangle $0 \leq \Im(s) \leq T$, $\sigma < \Re(s) \leq 1$, then $N(\sigma, T) = O(T)$, for any fixed $\sigma \geq ½$.

1919 Pólya conjectures that the summatory Liouville function,

$$L(x) := \sum_{n=1}^{x} \lambda(n),$$

where $\lambda(n)$ is the Liouville function, satisfies $L(x) \leq 0$ for $x \geq 2$. (This conjecture is proved incorrect by Haselgrove in 1958.)

1920 Carlson proves the density theorem.

Theorem 1 (The Density Theorem). *For any* $\varepsilon > 0$ *and* $\frac{1}{2} \leq \sigma \leq 1$, *we have* $N(\sigma, T) = O(T^{4\sigma(1-\sigma)+\varepsilon})$.

1922 Hardy and Littlewood show that the generalized Riemann hypothesis implies Goldbach's weak conjecture.

1923 Hardy and Littlewood prove that if the generalized Riemann hypothesis is true, then almost every even number is the sum of two primes. Specifically, if $E(N)$ denotes the number of even integers less than N that are not a sum of two primes, then
$E(N) \ll N^{\frac{1}{2}+\varepsilon}$.

1924 Franel and Landau discover an equivalence to the Riemann hypothesis involving Farey series. The details are not complicated, but are rather lengthy.

1925 Hutchinson calculates the first 138 zeros of $\zeta(s)$ on the critical line.

1928 Littlewood shows that the generalized Riemann hypothesis bounds $L(1, \chi_D)$ as

$$\frac{1}{\log\log D} \ll |L(1, \chi_D)| \ll \log\log D.$$

1932 Siegel analyzes Riemann's private (and public) papers. He finds (among other things) a formula for calculating values of $\zeta(s)$ that is more efficient than Euler–Maclaurin summation. The method is referred to as the Riemann–Siegel formula and is used in some form up to the present.

Siegel is credited with reinvigorating Riemann's most important results regarding $\zeta(s)$. In the words of Edwards, "It is indeed fortunate that Siegel's concept of scholarship derived from the older tradition of respect for the past rather than the contemporary style of novelty."

1934 Speiser shows that the Riemann hypothesis is equivalent to the nonvanishing of $\zeta'(s)$ in $0 < \sigma < \frac{1}{2}$.

1935 Titchmarsh calculates the first 1041 zeros of $\zeta(s)$ on the critical line.

1937 Vinogradov proves the following result related to Goldbach's conjecture without assuming any variant of the Riemann hypothesis.

Theorem 2. *Every sufficiently large odd number is a sum of three prime numbers.*

1940 Ingham shows that $N(\sigma, T) = O\left(T^{3\left(\frac{1-\sigma}{2-\sigma}\right)} \log^5 T\right)$. This is still the best known result for $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$.

1941 Weil proves that the Riemann hypothesis is true for function fields.

1942 Ingham publishes a paper building on the conjectures of Mertens and Pólya. He proves that not only do both conjectures imply the truth of the hypothesis, and the simplicity of the zeros, but they also imply a linear dependence between the imaginary parts of the zeros.

1942 Selberg proves that a positive proportion of the zeros of $\zeta(s)$ lie on the critical line.

1943 Alan Turing publishes two important developments. The first is an algorithm for computing $\zeta(s)$ (made obsolete by better estimates to the error terms in the Riemann–Siegel formula). The second is a method for calculating $N(T)$, which gives a tool for verifying the Riemann hypothesis up to a given height.

1948 Pál Turán shows that if for all N sufficiently large, the Nth partial sum of $\zeta(s)$ does not vanish for $\sigma > 1$, then the Riemann hypothesis follows.

1949 Selberg and Erdős, building on the work of Selberg, both find “elementary” proofs of the prime number theorem.

1951 Titchmarsh considers at length some consequences of a proof of the Riemann hypothesis. He considers sharper bounds for $\zeta(s)$, as well as the functions $S(T)$ and $S_1(T)$.

1951 Sometime before 1951, Titchmarsh found that the extended Riemann hypothesis can be applied in considering the problem of computing (or estimating) $\pi(x; k, l)$. He showed that if the extended Riemann hypothesis is true, then the least $p \equiv l \pmod{k}$ is less than $k^{2+\varepsilon}$, where $\varepsilon > 0$ is arbitrary and $k > k_0(\varepsilon)$.

1953 Alan Turing calculates the first 1104 zeros of $\zeta(s)$ on the critical line.

1955 Skewes bounds the first n such that $\pi(n) < \mathrm{Li}(n)$ fails. This bound is improved in the future, but retains the name “Skewes number.”

1955 Beurling finds the Nyman–Beurling equivalent form.

1956 Lehmer calculates the first 15,000 zeros of $\zeta(s)$ on the critical line, and later in the same year the first 25,000 zeros.

1958 Meller calculates the first 35,337 zeros of $\zeta(s)$ on the critical line.

1958 Haselgrove disproves George Pólya's conjecture.

1966 Lehman improves Skewes's bound.

1966 Lehman calculates the first 250,000 zeros of $\zeta(s)$ on the critical line.

1967 Hooley proves that Artin's conjecture holds under the assumption of the extended Riemann hypothesis. Artin's conjecture is the following:

Conjecture 3. *Every $a \in \mathbb{Z}$, where a is not square and $a \neq -1$, is a primitive root modulo p for infinitely many primes p.*

1968 Rosser, Yohe, and Schoenfeld calculate the first 3,500,000 zeros of $\zeta(s)$ on the critical line.

1968 Louis de Branges makes the first of his several attempts to prove the Riemann hypothesis. Other alleged proofs were offered in 1986, 1992, and 1994.

1973 Montgomery conjectures that the correlation for the zeros of the zeta function is

$$1 - \frac{\sin^2(\pi x)}{(\pi x)^2}.$$

1973 Chen proves that every sufficiently large even integer is a sum of a prime and number which is a product of at most two primes.

1974 The probabilistic Solovay–Strassen algorithm for primality testing is published. It can be made deterministic under the generalized Riemann hypothesis.

1975 The probabilistic Miller–Rabin algorithm for primality testing is published. It runs in polynomial time under the generalized Riemann hypothesis.

1977 Redheffer shows that the Riemann hypothesis is equivalent to the statement that

$$\det(R_n) = O(n^{\frac{1}{2}+\varepsilon})$$

for any $\varepsilon > 0$, where $R_n := [R_n(i,j)]$ is the $n \times n$ matrix with entries

$$R_n(i,j) = \begin{cases} 1 & \text{if } j = 1 \text{ or if } i \mid j \\ 0 & \text{otherwise.} \end{cases}$$

1977 Brent calculates the first 40,000,000 zeros of $\zeta(s)$ on the critical line.

1979 Brent calculates the first 81,000,001 zeros of $\zeta(s)$ on the critical line.

1982 Brent, van de Lune, te Riel, and Winter calculate the first 200,000,001 zeros of $\zeta(s)$ on the critical line.

1983 Van de Lune and te Riele calculate the first 300,000,001 zeros of $\zeta(s)$ on the critical line.

1983 Montgomery proves that the 1948 approach of Turán will not lead to a proof of the Riemann hypothesis. This is because for any positive $c < \frac{4}{\pi} - 1$, the Nth partial sum of $\zeta(s)$ has zeros in the half-plane $\sigma > 1 + c\frac{\log\log N}{\log N}$ [108].

1984 Ram Murty and Gupta prove that Artin's conjecture holds for infinitely many a without assuming any variant of the Riemann hypothesis.

1985 Odlyzko and te Riele prove that Mertens' conjecture is false. They speculate that, while not impossible, it is improbable that $M(n) = O(n^{1/2})$. The Riemann hypothesis is in fact equivalent to the conjecture $M(n) = O(n^{1/2+\varepsilon})$.

1986 Van de Lune, te Riele, and Winter calculate the first 1,500,000,001 zeros of $\zeta(s)$ on the critical line.

1986 Heath-Brown proves that Artin's conjecture fails for at most two primes.

1987 Te Riele lowers the Skewes number.

1988 Odlyzko and Schönhage publish an algorithm for calculating values of $\zeta(s)$. The Odlyzko–Schönhage algorithm is currently the most efficient algorithm for determining values $t \in \mathbb{R}$ for which $\zeta(1/2+it) = 0$. The algorithm computes the first n zeros of $\zeta(1/2+it)$ in $O(n^{1+\varepsilon})$ (as opposed to $O(n^{3/2})$ using previous methods).

1988 Barratt, Forcade, and Pollington formulate a graph-theoretic equivalent to the Riemann hypothesis through the use of Redheffer matrices.

1989 Odylzko computes 175 million consecutive zeros around $t = 10^{20}$.

1989 Conrey proves that more than 40% of the nontrivial zeros of $\zeta(s)$ lie on the critical line.

1993 Alcántara-Bode shows that the Riemann hypothesis is true if and only if the Hilbert-Schmidt integral operator on $L^2(0, 1)$ is injective.

1994 Verjovsky proves that the Riemann hypothesis is equivalent to a problem about the rate of convergence of certain discrete measures.

1995 Volchkov proves that the statement,

$$\int_0^\infty (1-12t^2)(1+4t^2)^{-3} \int_{\frac{1}{2}}^\infty \log|\zeta(\sigma+it)|d\sigma dt = \frac{\pi(3-\gamma)}{32}$$

is equivalent to the Riemann hypothesis, where γ is Euler's constant.

1995 Amoroso proves that the statement that $\zeta(s)$ does not vanish for $\Re(z) \geq \lambda + \varepsilon$ is equivalent to

$$\tilde{h}(F_N) \ll N^{\lambda+\varepsilon},$$

where

$$\tilde{h}(F_N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |F_N(e^{i\theta})| d\theta$$

and $F_N(z) = \prod_{n \leq N} \Phi_n(z)$, where $\Phi_n(z)$ denotes the nth cyclotomic polynomial, and $\log^+(x) = \max(0, \log x)$.

1997 Hardy and Littlewood's 1922 result concerning Goldbach's conjecture is improved by Deshouillers, Effinger, te Riele, and Zinoviev. They prove the following result.

Theorem 4. *Assuming the generalized Riemann hypothesis, every odd number greater than 5 can be expressed as a sum of three prime numbers*.

2000 Conrey and Li argue that the approach used by de Branges cannot lead to proof of the Riemann hypothesis.

2000 Bays and Hudson lower the Skewes number.

2000 The Riemann hypothesis is named by the Clay Mathematics Institute as one of seven "Millennium Prize Problems." The solution to each problem is worth one million US dollars.

2001 Van de Lune calculates the first 10,000,000,000 zeros of $\zeta(s)$ on the critical line.

2004 Wedeniwski calculates the first 900,000,000,000 zeros of $\zeta(s)$ on the critical line.

2004 Gourdon calculates the first 10,000,000,000,000 zeros of $\zeta(s)$ on the critical line.

2016 LMFDB (The L-functions and Modular Forms Database) is founded. LMFDB is an open-source of zeta-zero values.

2018 Sir Michael Atiyah announces the proof of Riemann hypothesis by method of contradiction.

2020 Platt & Trudgian calculate the first 12363153437138 up to height 3000175332800 zeros of $\zeta(s)$ on the critical line. They also verified the work of Gourdon (2004) and others.

2020 LMFDB calculates the first 10^{36} zeros of $\zeta(s)$ on the critical line.

*The above timeline was prepared by Peter Borwein team, and updated by M.S.